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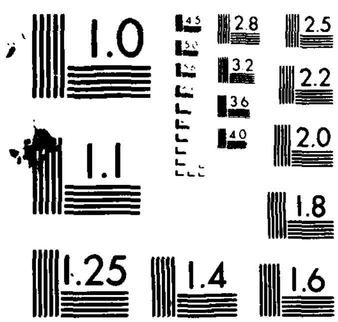

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 82-0261	2. GOVT ACCESSION NO. AD-A113077	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SUMMABILITY METHODS FOR DIVERGENT INTEGRALS AND THEIR APPLICATION TO SINGULAR STURM-LIOUVILLE EXPANSIONS		5. TYPE OF REPORT & PERIOD COVERED Final
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Louise A. Raphael		8. CONTRACT OR GRANT NUMBER(s) AFOSR-81-0075
9. PERFORMING ORGANIZATION NAME AND ADDRESS Clark College Department of Mathematics Atlanta, GA 30314		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A4
11. CONTROLLING OFFICE NAME AND ADDRESS AFOSR/NM Bolling AFB, DC 20332		12. REPORT DATE 26 Feb 82
		13. NUMBER OF PAGES 26
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) unclassified
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

These two papers continue the investigation of the Stieltjes summability method, its generalizations and their application to summing singular Sturm-Liouville (S-L) eigenfunctions.

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unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AFOSR-TR- 82-0261

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
MINI-GRANT [redacted] Final REPORT

on

SUMMABILITY METHODS FOR DIVERGENT INTEGRALS
AND THEIR APPLICATION TO SINGULAR
STURM-LIOUVILLE EXPANSIONS

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Distribution/	
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AFOSR-81-0075

Control Number: 80-NM-279

Home Institution: Clark College
Department of Mathematics
Atlanta, Georgia 30314

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Washington, D.C. 20059

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Principal Investigator: Louise A. Raphael
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AMS(MOS) subject classifications (1970). Primary 34B25; Secondary 40A10, 42A76. Key words and phrases. Sturm-Liouville expansions, stable summability, singular.

82 04 06 006

ABSTRACT. These two papers continue the investigation of the Stieltjes summability method, its generalizations and their application to summing singular Sturm-Liouville (S-L) eigenfunctions.

Paper 1: Given a general singular S-L system on a semi-infinite domain possessing a discrete negative and continuous positive spectrum, the problem of inverting the generalized Fourier transform of $L^p(0, \infty)$ $(1 \le p < \infty)$ functions is considered. The generalized S-L eigenfunction expansions of $L^p(0, \infty)$ $(1 \le p < \infty)$ functions f are shown to be Stieltjes summable to f with respect to the $L^p(0, \infty)$ norm for $1 \le p < \infty$ and pointwise on the Lebesgue set of f. As an immediate application we see that the Stieltjes summability means of eigenfunction expansions with perturbed coefficients converge pointwise to the original function.

Paper 2: (Joint with Professor Mark Kon of Boston University) This paper unifies and generalizes a number of results in classical summability theory for regular and singular S-L expansions. As this generalized summability method is pointwise stable, it has practical application to the area of ill-posed problems. Namely, it sums S-L eigenfunction expansions with perturbed coefficients to the original function.

1. <u>Introduction</u>: The general nature of this AFOSR mini-grant was to further the investigation of the Stieltjes summability method for divergent integrals and its application to the summation of singular Sturm-Liouville eigenfunction expansions with a continuous spectral component. This work is a continuation of research done under Dr. Paul Nikolai at the Flight Dynamics Laboratory, Wright Patterson Air Force Base during the summer of 1980. Much of the new research results were obtained during the summer of 1981 when the investigator visited the Massachusetts Institute of Technology and worked jointly with Professor Mark Kon of Boston University. The completion of the research and the writing of the two papers was done under an Army Research Office grant DAA-G29-81-G-0011.

Three major goals of the research has been to study

(1) the properties of Stieltjes kernel associated with the Stieltjes summability method;

(2) the summing of eigenfunction expansions

  (a) pointwise and with respect to the $L^p(0, \infty)$ $(1 \le p < \infty)$ norm, and

  (b) by developing summation methods stable under perturbation of the expansion coefficients; and

(3) generalization of classical summability results for regular and singular S-L eigenfunction expansions.

These results were presented, upon invitation, at the special session Summability Theory of the annual American Mathematical Society meeting in Cincinnati in January 1982.

2. Review of Previous Related Research. A summative technical description of the previous research work now follows.

a) Stieltjes summability:

The Stieltjes summability method for divergent integrals and some of its properties were developed by Raphael during the period 1977-78 [1]. The divergent integral $\int_0^\infty f(x)dx$ is said to be Stieltjes summable order w with respect to g(x) if $\lim_{\alpha\to 0} \int_0^\infty \frac{f(x)dx}{[1 + \alpha g(x)]^w}$ exists. The function g(x) is assumed monotone increasing to infinity. Stieltjes summability was compared with Cesaro and Abelian summability, expanding on some previous work of Bromwich [2] and Moore [3] who treated some of the comparison questions under a different formulation. Stieltjes summability has been applied to discrete series by Hille [4] under the name resolvent summability, and recently by Judak [5] under the name T-summation.

In [1] the Stieltjes summability method with g(x) = x and w = 1 was applied to the stable pointwise summation of expansions in eigenfunctions of a singular Sturm-Liouville system on the half-line: $u'' - q(x)u = -\lambda u$; $u(0) = 0$, $u(\infty) < \infty$ where q(x) is continuous, bounded and in $L_1[0, \infty)$. This system has a continuous spectral component of $\{\lambda > 0\}$ and a bounded, discrete non-positive spectral component. The generalized Fourier transform and inverse transform are written in the form:

$$F(\lambda) \sim \int_0^\infty f(x)u(x, \lambda)dx, \quad f(x) \sim \int_{-b}^\infty F(\lambda)u(x, \lambda)d\rho(\lambda)$$

where $f(x) \in L_2[0, \infty)$, $F(\lambda)$ is the generalized Fourier transform of f, $\rho(\lambda)$ is the spectral function, $-b = \inf(\lambda)$, and the convergence, denoted by ~ is in the L_2 norm. (The latter integral is what is meant by "the eigenfunction expansion of f".)

The Stieltjes means of the inverse transform are given by

$$S_\alpha(F;\, x) = \int_{-b}^{\infty} \frac{F(\lambda)u(x,\, \lambda)}{1 + \alpha\lambda}\, d\rho(\lambda)$$

The following theorem, proved in [1], provides a stable summation method for the inverse transform based on the Stieltjes means:

Theorem A: Let $\{F_\gamma(\lambda)\}$ denote a net of approximations of $F(\lambda)$ such that $|F_\gamma - F|_{2,\rho} \leq \gamma$. If the summation parameter α is scaled with γ so that $\alpha = k\gamma^2$ for some $k > 0$, then $S_\alpha(F_\gamma;\, x) \to f(x)$ as $\gamma \to 0$ if x is a continuity point of f.

An immediate corollary is that the inverse transform is Stieltjes summable to f at continuity points. (Take $F_\gamma = F$.) Theorem A was proved by applying the regularization method of Tikhonov, who used it in [6] to prove an analogous result for regular Sturm-Liouville expansions.

In [7] Diamond, Kon and Raphael generalized Theorem A to Lebesgue points with a sharper scaling.

b) A general class of stable summation methods for singular Sturm-Liouville expansions on the half-line (see [8]):

The Sturm-Liouville system considered in [8] is slightly more general than in [1]: $u'' - q(x)u = -\lambda u$, $u(0)\cos\beta + u'(0)\sin\beta = 0$, $u(\infty) < \infty$, where $q(x)$ is continuous and in $L_1[0, \infty)$. General summation methods of the form

$$S_\alpha(F;\, x) = \int_{-b}^{\infty} F(\lambda)\phi(\alpha\lambda)u(x,\, \lambda)\,d\rho(\lambda)$$

are studied, where the summator function ϕ satisfies $\phi(0) = 1$ and is bounded. In particular, stable summation methods based on the summability means S_α are developed.

The main results are as follows:

Theorem B: Suppose the following two conditions hold:

a) $S_\alpha(F; x) \to f(x)$ as $\alpha \to 0$ (i.e. the inverse transform is ϕ-summable to f at x)

b) $\int_0^\infty \frac{\phi(t)}{t^{1/2}} dt < \infty$.

If α is scaled with γ so that $\gamma/\alpha^{1/4} \to 0$ and $\alpha \to 0$ as $\gamma \to 0$, then $S_\alpha(F_\gamma; x) \to f(x)$ as $\gamma \to 0$ (i.e., S_α is a stable summation method). On the other hand, if $\gamma/\alpha^{1/4} \not\longrightarrow 0$ as $\gamma \to 0$ then there exist $\{F_\gamma\}$ such that $S_\alpha(F_\gamma; x) \not\longrightarrow f(x)$ as $\gamma \to 0$ (i.e., S_α is not a stable summation method).

The proof of Theorem B relies on results from the spectral theory of singular Sturm-Liouville systems as developed in [9] and [10]. Theorem B was motivated by a paper of Krukovski, [11], who developed the analogous class of stable summation methods and sharp scaling requirements for (discrete) expansions in eigenfunctions of the N-dimensional Laplacian.

Theorem B is applied in [8] to the Stieltjes and Riesz summability methods, for which a priori summability of the inverse transform is known. The Stieltjes summability result of [1] is extended to include the case of Lebesgue points.

c) Absolute summability methods for divergent integrals. While working at the Flight Dynamics Laboratory, Wright-Patterson Air Force Base, Raphael considered applying Stieltjes summability methods to divergent integrals on $[0, \infty)$, whose summability means are of bounded variation. With Diamond, she proved in [12] inclusion theorems relating the absolute summability of divergent integrals for two Abelian summability and the Stieltjes summability methods. These results were presented at the annual meeting of the American Mathematical Society in San Francisco in January 1981. The preprint based on this work has been submitted for publication to the Canadian Bulletin of Mathematics.

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NEW MULTIPLIER METHODS FOR SUMMING CLASSICAL EIGENFUNCTION EXPANSIONS

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* Partially supported by the NSF under grant MCS-8003407

New Multiplier Methods for Summing Classical Eigenfunction Expansions

§1. Introduction

The purpose of this paper is to unify and generalize a number of results in classical summability theory for Sturm-Liouville eigenfunction expansions (see, e.g., [7],[8]), and in summability for singular Sturm-Liouville expansions. We will show that summability of such expansions with analytic summator functions, which has been proved in a number of individual cases, is actually a consequence of so-called resolvent summability and a "superposition principle" for summator functions.

Classically, summability theory of eigenfunction expansions has dealt with the application of various *summability methods* (e.g. Abel, Riesz) to expansions in eigenfunctions $\{u_n(x)\}$ of a self-adjoint boundary value (typically Sturm-Liouville) problem. A *summability method* in its more modern sense is a net of linear operators $\{\phi_\epsilon\}_{\epsilon \in S \subset \mathbb{C}}$ which forms an approximate identity in an appropriate function space. In the classical case, we make the restriction $\phi_\epsilon = \phi(\epsilon A)$, where A is a self-adjoint differential operator, ϕ is a function on $\mathbb{R}$, and $\phi(\epsilon A)$ is defined by operator calculus. Then if $f(x) \in \mathcal{D}(A)$ and

$$f(x) \sim \sum a_n u_n(x)$$

is an expansion (continuous or discrete) in eigenfunctions of A, then

$$\phi(\epsilon A)f(x) \sim \sum a_n \phi(\epsilon n) u_n(x).$$

The function f is *ϕ-summable* in a given topology if $\phi(\epsilon A)f \underset{\epsilon \to 0}{\longrightarrow} f$; the topology may be one of convergence pointwise or in some function space, generally L^p. In this role ϕ is a *summator function*.

Throughout this paper $A = -\frac{d^2}{dx^2} + q$ will be a Sturm-Liouville operator. The success of the approach used here is based on superposition principle: if a function f is ϕ_1-and ϕ_2-summable, then it is $\alpha_1\phi + \alpha_2\phi_2$-summable, for $\alpha_1, \alpha_2 \in \mathbf{C}$. If one can obtain a sum or integral representation of a summator function in terms of functions with respect to which f is summable, then certain regularity properties of the representation will suffice to prove summability for this function. Correspondingly, since the integral kernel of a summability method depends linearly on the summator function, this principle holds for the kernels of summability methods, which is the central fact used in this paper.

Our procedure begins with a proof that when $\phi(\epsilon\lambda) = (1 + \epsilon\lambda)^{-1}$ for ϵ in a complex domain, then L^2 functions are ϕ-summable in L^p and pointwise in fairly general one dimensional situations. In this case ϕ-summability is known as *resolvent summability*. The proof is accomplished by estimation of the integral kernel of $\phi(\epsilon A)$, which is the Green function of $A + \frac{1}{\epsilon}$. This generalizes known results on resolvent summability [6,2]. From this, expressions for kernels for more general summability methods are obtained through contour integration of the resolvent kernel. It is then proved that summability holds if ϕ is an analytic function satisfying certain minimal constraints. This extends a body of results (see [6], [7], [8]) which deal with specific summability methods.

The proofs here will be carried out for the class of singular continuous spectrum Sturm-Liouville expansions treated in [1,2]. They carry over (in some cases more simply) to regular expansions on finite intervals.

We now present the problem to be considered in more detail. Let S-L denote the singular Sturm-Liouville system

$$Au(x,\lambda) \equiv \left(-\frac{d^2}{dx^2} + q(x)\right)u(x,\lambda) = \lambda u(x,\lambda);$$

$$u(0,\lambda)\cos\beta + u'(0,\lambda)\sin\beta = 0; \qquad u(\infty,\lambda) < \infty, \tag{1}$$

with $q(x) \in L^1[0,\infty)$ real valued, continuous, and bounded, and $\beta \in [0, 2\pi)$. Let $u(x,\lambda)$ be normalized by

$$u(0,\lambda) = \sin\beta, \quad u'(0,\lambda) = -\cos\beta \tag{2}$$

The spectrum of S-L is bounded from below (say by $-b$), discrete for $\lambda \leq 0$, and continuous for $\lambda > 0$ (see [4], Chap. 3). If $f \in L^2[0,\infty)$, the S-L expansion of f is given by

$$f \sim \int_{-b}^{\infty} F(\lambda)u(x,\lambda)d\rho(\lambda), \tag{3}$$

where

$$F(\lambda) \sim \int_0^{\infty} f(x)u(x,\lambda)dx.$$

Above, ρ is the spectral function associated with the system, and F, the generalized Fourier transform of f, is in $L^{2,\rho}$, i.e.,

$$\int_{-b}^{\infty} F^2(\lambda)d\rho(\lambda) < \infty.$$

Throughout this paper, $\sim$ denotes convergence in L^2 as the upper limit of integration becomes infinite.

We remark thet the restriction on the potential q in the S-L operator and on the function f are made in order that the spectral representations given here make sense.

If one takes the summation method to be defined by the integral operator coresponding to $\phi(\epsilon A)$, then the main results of this paper hold for $f \in L^p$ and a much less restricted q.

The pointwise convergence of the integral in (3) is well known to be a delicate question. We define the *summability means*

$$\phi(\epsilon A) \equiv \int_{-b}^{\infty} \phi_\epsilon(\lambda) F(\lambda) u(x, \lambda) d\rho(\lambda), \tag{4}$$

where ϕ: C $\mapsto$ C is a *summator function*, with the property $\phi(0) = 1$.

In [2], the expansion (3) is proved to be ϕ-summable at x when $\phi(\epsilon\lambda) = (1 + \epsilon\lambda)^{-1}$ ($\epsilon >$ 0) and x is a Lebesgue point of f, i.e.,

$$\lim_{\eta \to 0} \frac{1}{2\eta} \int_{x-\eta}^{x+\eta} f(x + \eta) d\eta = f(x). \tag{5}$$

We next prove this for ϵ complex.

We now mention some conventions. Unless otherwise specified, functions will be defined on $[0, \infty)$. By *contour* we mean a rectifiable curve. If $C \subset$ C, then $\epsilon C = \{\epsilon c : c \in C\}$. $\sigma(A)$ and $\mathcal{D}(A)$ denote the spectrum and the domain of the operator A.

§2. Resolvent Summability in the Complex Plane.

Our starting point is a straightforward calculation of the Green function G_α^* of the modified operator $-\frac{d^2}{dx^2} + \frac{1}{\alpha}$, Re$\alpha \notin$ R$^-$, with boundary conditions (2):

$$G_\alpha^*(x, x') = \frac{\sqrt{\alpha}}{(\sin\beta - \cos\beta)} e^{-\frac{x_>}{\sqrt{\alpha}}} \left\{ \cos\beta \sinh\frac{x_<}{\sqrt{\alpha}} - \sin\beta \cosh\frac{x_<}{\sqrt{\alpha}} \right\}, \tag{6}$$

where $x_> = \max\{x, x'\}$ and $x_< = \min\{x, x'\}$ and we assume $\sin\beta \neq \cos\beta$; one checks easily that the boundary conditions at 0 and ∞ are satisfied.

Let G_α be the Green function for $A_\alpha = -\frac{d^2}{dx^2} + q(x) + \frac{1}{\alpha}$, with $q(x)$ as in S-L, and α sufficiently small that A_α is positive. The distributional equations

$$\left(-\frac{d^2}{dx^2} + \frac{1}{\alpha}\right)G^*_\alpha(x, x') = \delta_{x'}(x)$$

$$\left(-\frac{d^2}{dx^2} + q(x) + \frac{1}{\alpha}\right)G_\alpha(x, x') = \delta_{x'}(x) \tag{7}$$

yield (after subtraction)

$$G_\alpha(x, x') - G^*_\alpha(x, x') = -\int_0^\infty G^*_\alpha(x, x'')q(x'')G_\alpha(x'', x')dx'' \tag{8}$$

The integral on the right is easily shown to converge absolutely via the boundary conditions (1) on G_α.

It will later be shown that $\phi(\alpha A)$ as an operator on $L^2(0, \infty)$ has kernel $\frac{1}{\alpha}G_\alpha(x, x')$, if $\phi(\alpha\lambda) = (1 + \alpha\lambda)^{-1}$. With this motivation we now prove

THEOREM 1: *Let $f \in L^p[0, \infty)$ $(1 \leq p \leq \infty)$. Then for any $\gamma > 0$ and α in the sector $\Omega_\gamma \equiv \{\alpha : |\arg\alpha| \leq \pi - \gamma\}$,*

$$\frac{1}{\alpha}\int_0^\infty G_\alpha(x, x')f(x')dx' \underset{\alpha\to 0}{\to} f(x)$$

in L^p (for $1 \leq p < \infty$), and pointwise on in the Lebesgue set of f.

The proof proceeds by estimates on G_α and the use of a theorem from harmonic analysis on R^n. We define $D_r = \{\alpha \in \mathrm{C} : |\alpha| \leq r\}$.

LEMMA 1.1: *The L^1-norm $\|\frac{1}{\alpha}G_\alpha(x, x')\|_{1,x}$ (respectively, $\|\frac{1}{\alpha}G_\alpha(x, x')\|_{1,x'}$) is uniformly bounded in x' (x) and $\alpha \in \Omega_\gamma \cap D_r$ for r sufficiently small. Furthermore,*

$$\frac{1}{\alpha}\|G^*_\alpha(x, x') - G_\alpha(x, x')\|_{1,x\,(x')} = 0(|\alpha|)$$

uniformly in x' (x) *as* $\alpha \to 0$ *in* Ω_γ.

Proof: The first statement can be checked explicitly for $G^*_\alpha(x,x')$. By (8) and the Minkowski inequality for integrals

$$\begin{aligned}
\|\frac{1}{\alpha}G_\alpha(x,x')\|_{1,x} - \|\frac{1}{\alpha}G^*_\alpha(x,x')\|_{1,x} &\le \|\frac{1}{\alpha}(G^*_\alpha(x,x') - G_\alpha(x,x'))\|_{1,x} \\
&\le \|q\|_\infty \int_0^\infty \|\frac{1}{\alpha}G^*_\alpha(x,x'')\|_{1,x}|G_\alpha(x'',x')|dx'' \\
&\le C\|q\|_\infty \int_0^\infty |G_\alpha(x'',x')|dx'' \\
&= C\|q\|_\infty \|G_\alpha(x,x')\|_{1,x},
\end{aligned} \tag{9}$$

where C is such that $\|\frac{1}{\alpha}G^*_\alpha(x,x')\|_{1,x} \le C \quad (\alpha \in \Omega_\gamma)$. Hence

$$\|\frac{1}{\alpha}G_\alpha(x,x')\|_{1,x} - C \le C\|q\|_\infty\|G_\alpha(x,x')\|_{1,x},$$

and

$$\|\frac{1}{\alpha}G_\alpha(x,x')\|_{1,x} \le \frac{C}{1-\alpha C\|q\|_\infty} \tag{10}$$

Similarly, since

$$G_\alpha(x,x') - G^*_\alpha(x,x') = -\int_0^\infty G_\alpha(x,x'')q(x'')G^*_\alpha(x'',x')dx'',$$

we have

$$\|\frac{1}{\alpha}G_\alpha(x,x')\|_{1,x'} \le \frac{C'}{1-\alpha C'\|q\|_\infty},$$

where C' satisfies $\|\frac{1}{\alpha}G^*_\alpha(x,x')\|_{1,x'} \le C', \ \alpha \in \Omega_\gamma$.

By (9) and (10)

$$\frac{1}{\alpha}\|G^*_\alpha(x,x') - G_\alpha(x,x')\|_{1,x} = 0(\alpha),$$

the proof being the same when $x \leftrightarrow x'$. ∎

LEMMA 1.2: *As* $\alpha \to 0$ *in* Ω_γ,

$$\|\frac{1}{\alpha}G^*_\alpha(x,x') - \frac{1}{\alpha}G_\alpha(x,x')\|_\infty = 0\left(\sqrt{|\alpha|}\right)$$

uniformly in arg α.

Proof: By (6), $\|G^*_\alpha(x,x')\|_\infty \leq \frac{2\sqrt{\alpha}}{\sin\beta - \cos\beta}$. Hence by (8) and (10)

$$\begin{aligned}\frac{1}{\alpha}|G^*_\alpha(x,x') - G_\alpha(x,x')| &\leq \frac{1}{\alpha}\|G^*_\alpha(x,x')\|_\infty\|q\|_\infty\|G_\alpha(x,x')\|_{1,x} \\ &\leq \frac{2\|q\|_\infty}{(\sin\beta - \cos\beta)} \cdot \frac{\sqrt{\alpha}C}{1 - \alpha C\|q\|_\infty}. \blacksquare\end{aligned} \tag{11}$$

Proof of Theorem 1: We first prove the Theorem for G^*_α. By (6),

$$\frac{1}{\alpha}G^*_\alpha(x,x') = \frac{1}{\alpha}G^{(1)}_\alpha(x,x') - \frac{1}{\alpha}G^{(2)}_\alpha(x,x'), \tag{12}$$

where

$$G^{(1)}_\alpha(x,x') = \sqrt{\alpha}\frac{e^{\frac{-|x-x'|}{\sqrt{\alpha}}}}{2}, \qquad G^{(2)}_\alpha(x,x') = \sqrt{\alpha}\frac{\sin\beta + \cos\beta}{2(\sin\beta - \cos\beta)}e^{-\frac{|x+x'|}{\sqrt{\alpha}}}.$$

Note that both $\frac{1}{\alpha}G^{(1)}_\alpha(x,x')$ and $\frac{1}{\alpha}G^{(2)}_\alpha(-x,x')$ are nets of L^1-dilations of radially decreasing convolution kernels on $\mathbf{R}$. If we extend $f(x)$ to be 0 for $x < 0$, the conclusion of the Theorem for $G^{(1)}_\alpha$ is then the statement of a well-known result of harmonic analysis (see [5], Ch. 1). It thus remains to prove that

$$\int_0^\infty \frac{1}{\alpha}G^{(2)}_\alpha(x,x')f(x')dx' \underset{\alpha\to 0}{\to} 0 \tag{13}$$

in L^p $(1 \leq p < \infty)$, and on the Lebesgue set of f. But this is clear in view of the above-stated result and the fact that the replacement $x \to -x$ changes $G^{(2)}_\alpha(x,x')$ into a convolution kernel , and $f(x)$ into a function which is 0 on $\mathbf{R}^+$. To complete the proof, we define the mixed p,q norm by

$$\|F(x,x')\|_{p,q} = \left(\int_0^\infty \left(\int_0^\infty |F(x,x')|^p dx \right)^{\frac{q}{p}} dx' \right)^{\frac{1}{q}}. \tag{14}$$

By Lemmas 1.1 and 1.2 and standard L^p interpolation theory, $\|\frac{1}{\alpha}G_\alpha(x,x')\|_{p,x'}$ and $\|\frac{1}{\alpha}G_\alpha(x,x')\|_{p,x}$ are uniformly bounded in $\alpha \in \Omega_\gamma$ for α small. Hence by (8) and the Minkowski inequality,

$$\begin{aligned}\frac{1}{\alpha}\|G^*_\alpha(x,x') - G_\alpha(x,x')\|_{p,q} &\leq \int_0^\infty \frac{1}{\alpha}\|G^*_\alpha(x,x')\|_{p,x}|q(x'')|\|G_\alpha(x'',x')\|_{q,x'}dx'' \\ &\leq \sup_{x''}\left\{\|\frac{1}{\alpha}G^*_\alpha(x,x'')\|_{p,x}\|G_\alpha(x'',x')\|_{q,x'}\right\}\|q\|_1 \underset{\alpha\to 0}{\longrightarrow} 0,\end{aligned} \tag{15}$$

for $\alpha \in \Omega_\gamma$.

Let $f \in L^p$ $(1 \leq p \leq \infty)$, and $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\left| \int_0^\infty \frac{1}{\alpha}(G^*_\alpha(x,x') - G_\alpha(x,x'))f(x')dx' \right| \leq \frac{1}{\alpha}\|G^*_\alpha(x,x') - G_\alpha(x,x')\|_{p',x'}\|f\|_p \to 0$$

as $\alpha \to 0$ in Ω_γ, by Lemmas 1.1 and 1.2.

In addition, if $1 \leq p < \infty$,

$$\|\int_0^\infty \frac{1}{\alpha}(G^*_\alpha(x,x') - G_\alpha(x,x'))f(x')dx'\|_p \leq \frac{1}{\alpha}\|G^*_\alpha(x,x') - G_\alpha(x,x')\|_{p,p'}\|f\|_p \underset{\alpha\to 0}{\longrightarrow} 0,$$

by Hölder's inequality. This finishes the proof.∎

§3. Summability for Analytic Multipliers

In this section we develop a representation of the kernel of a summability method $\phi(\epsilon A)$ for ϕ analytic, via contour integration of the Green function G_α. We continue with the definitions and assumptions in §1.

We require the following bounds on the eigenfunctions $u(x,\lambda)$ and spectral function ρ of S-L. These follow immediately from well known estimates ([4], Ch. 3).

PROPOSITION 2: *As $\lambda \to \infty$, $u(x,\lambda)$ and $\rho(\lambda)$ satisfy (a)*

$$u(x,\lambda) = \frac{-\cos\beta}{\sqrt{\lambda}} \sin\sqrt{\lambda}x + 0\left(\frac{1}{\lambda}\right) \tag{16}$$

$$\rho'(\lambda) = \frac{\sqrt{\lambda}}{\pi\cos^2\beta} + 0(1)$$

if $\sin\beta = 0$, and *(b)*

$$u(x,\lambda) = \sin\beta\cos\sqrt{\lambda}x + 0\left(\frac{1}{\sqrt{\lambda}}\right)$$

$$\rho'(\lambda) = \frac{1}{\pi\sqrt{\lambda}\sin^2\beta} + 0\left(\frac{1}{\lambda}\right)$$

if $\sin\beta \neq 0$.

Let $\{\epsilon\}_{\epsilon\in C}$, $S \subset C$ be a net of numbers converging to 0. Let $\phi(z)$ be an analytic function in a simply connected open domain D, such that $\frac{1}{\epsilon}D \supset \sigma(A)$ for ϵ sufficiently small. Let C be a positively oriented simple closed contour through ∞ such that for ϵ sufficiently small:

(i) $\sigma(A) \subset C \subset B_\gamma \equiv \{z: \ |\arg z| \leq \gamma\}$ for some $\gamma > 0$

(ii) $D \supset \epsilon C$

(iii) $\int_C \left|\frac{\phi(\epsilon z)}{z}\right| dz < \infty$

(iv) For w in C, $\phi(\epsilon w) = \int_C \frac{\phi(\epsilon z)}{z-w} dz$.

These four conditions will be denoted by [C].

The following lemma follows directly from the theory of operator calculi in Hilbert space. We include an explicit proof for completeness.

LEMMA 3.1: *Let $f \in L^2[0,\infty)$. Then*

$$\phi(\epsilon A)f \equiv \int_{-b}^{\infty} F(\lambda)u(x,\lambda)\phi(\epsilon\lambda)d\rho(\lambda) = \frac{1}{2\pi i}\int_C \frac{\phi(\epsilon z)}{z}\left(1-\frac{A}{z}\right)^{-1} f(x)dz \qquad (17)$$

where both sides are considered as integrals of L^2-valued functions.

Proof: By (iii) and (iv) $\phi(\epsilon z)$ is bounded on $\mathbf{R}^+$; hence $\phi(\epsilon A)$ is a bounded operator on $L^2[0,\infty)$. On the other hand, $(1-\frac{1}{z}A)^{-1}$ has kernel $-zG_{-\frac{1}{z}}(x,x')$; by Theorem 1, $(1-\frac{1}{z}A)^{-1}$ is uniformly bounded from L^2 to L^2 for $z \in C$. Hence by (iii), the right side of (17) is bounded as an operator on f.

It thus suffices to show that the left and right sides of (17) agree on a dense subset of $L^2[0,\infty)$. To this end, let $f \in L^2[0,\infty)$, and $F(\lambda) \in L^{1,\rho}$. Then

$$\begin{aligned}\phi(\epsilon A)f &= \int_{-b}^{\infty} F(\lambda)u(x,\lambda)\phi(\epsilon\lambda)d\rho(\lambda)\\ &= \frac{1}{2\pi i}\int_{-b}^{\infty}\int_C \frac{\phi(\epsilon z)}{z}\left(1-\frac{\lambda}{z}\right)^{-1} F(\lambda)u(x,\lambda)dz\,d\rho(\lambda)\\ &= \frac{1}{2\pi i}\int_C\int_{-b}^{\infty} \frac{\phi(\epsilon z)}{z}\left(1-\frac{\lambda}{z}\right)^{-1} F(\lambda)u(x,\lambda)d\rho(\lambda)dz\\ &= \frac{1}{2\pi i}\int_C \frac{\phi(\epsilon z)}{z}\left(1-\frac{A}{z}\right)^{-1} f(x)dz.\end{aligned} \qquad (18)$$

The interchange of integration is justified by

$$\int_C\int_{-b}^{\infty}\left|\frac{\phi(\epsilon z)}{z}\right|\left|1-\frac{\lambda}{z}\right|^{-1}|F(\lambda)||u(x,\lambda)||d\rho||dz|$$

$$\leq \left(\sup_{\lambda\in\sigma(A),z\in C}\left|1-\frac{\lambda}{z}\right|^{-1}\right)||F(\lambda)||_{1,\rho}||u(x,\lambda)||_{\infty,\lambda}\int_C\left|\frac{\phi(\epsilon z)}{z}\right||dz|;$$

the boundedness of $u(x,\lambda)$ in λ follows from continuity in λ and Proposition 2.∎

LEMMA 3.2: *If $f \in L^2(0,\infty]$ and*

$$\int_{-b}^{\infty}\frac{\phi^2(\epsilon\lambda)}{\sqrt{\lambda}}d\lambda < \infty \qquad (19)$$

then $\phi(\epsilon A)f(x)$ is continuous.

Proof: By the continuity of $u(x,\lambda)$, it suffices to show

$$\int_M^\infty F(\lambda)u(x,\lambda)\phi(\epsilon\lambda)d\rho(\lambda) \underset{M\to\infty}{\to} 0 \tag{20}$$

uniformly on compact x-sets. (20) is bounded by

$$||F||_{2,\rho}\int_M^\infty u^2(x,\lambda)\phi^2(\epsilon\lambda)d\rho(\lambda) \underset{M\to\infty}{\to} 0,$$

the convergence being uniform on bounded x-intervals, by Proposition 2. ∎

LEMMA 3.3: *If*

$$\int_C \left|\frac{\phi(\epsilon z)}{z}\right||dz| < \infty, \tag{21}$$

then $\int_C \frac{\phi(\epsilon z)}{z}(1-\frac{A}{z})^{-1}f(x)dz$ is continuous in x if $f \in L^2$.

Proof: We have

$$\int_C \frac{\phi(\epsilon z)}{z}\left(1-\frac{A}{z}\right)^{-1} f(x)dz = \int_C\int_0^\infty \frac{\phi(z)}{z}\left(-zG_{-\frac{1}{z}}(x,x')\right)f(x')dx'dz; \tag{22}$$

$G_{-\frac{1}{z}}(x,x')$ is continuous in x, and the right side of (22) converges absolutely:

$$\int_C\int_0^\infty \left|\frac{\phi(\epsilon z)}{z}\right|\left|zG_{-\frac{1}{z}}(x,x')\right||f(x')|dx'|dz| \le ||f||_2||zG_{-\frac{1}{z}}(x,x')||_{2,x'}\int_C\left|\frac{\phi(\epsilon z)}{z}\right||dz|, \tag{23}$$

which is uniformly bounded for bounded x. ∎

From Lemmas 3.1-3.3, we conclude:

PROPOSITION 3: If (19) and (21) hold, then for $f \in L^2$, then $\phi(\epsilon A)f(x)$ is continuous, and for all x,

$$\phi(\epsilon A)f(x) = \frac{1}{2\pi i}\int_C \frac{\phi(\epsilon z)}{z}\left(1-\frac{A}{z}\right)^{-1} f(x)dz. \tag{24}$$

We will now obtain a representation of the kernel for $\phi(\epsilon A)$, from which summability properties are deducible.

THEOREM 4: *If A is the S-L operator of §1, and ϕ is as above, then the summability method $\phi(\epsilon\lambda)$ has kernel*

$$K_\epsilon(x,x') = -\frac{1}{2\pi i}\int_C \phi(\epsilon z) G_{-\frac{1}{z}}(x,x') \; dz \tag{25}$$

for ϵ sufficiently small where C is any contour satisfying [C].

Proof: If $f \in L^2$ we have by (23),

$$\begin{aligned} \phi(\epsilon A)f &= \frac{1}{2\pi i}\int_C \phi(\epsilon z)(z-A)^{-1} f(x)dz \\ &= -\frac{1}{2\pi i}\int_C \phi(\epsilon z)\int_0^\infty G_{-\frac{1}{z}}(x,x')f(x')dx'dz \\ &= \int_0^\infty K_\epsilon(x,x')f(x')dx'. \blacksquare \end{aligned} \tag{26}$$

Note that $K_\epsilon(x,x')$ is finite when $x \neq x'$ for all ϵ such that (iii) holds, by (6) and Lemma 1.2.

We now establish properties of K_ϵ which suffice to prove ϕ-summability for S-L eigenfunction expansions. In allowing $\epsilon \to 0$ in C, we assume certain constraints which are required in order that K_ϵ remain well defined. We continue with the assumptions and notation of the previous sections.

THEOREM 5: *Let $\{\epsilon\}_{\epsilon \in S}$ be a net of complex numbers, and ϕ be analytic in a simply connected domain D such that there exists a C satisfying* [C]. *Let $f \in L^2(\mathrm{R}^+)$. Then f is ϕ-summable as $\epsilon \xrightarrow{s} 0$ on its Lebesgue set and in L^p for all p such that $f \in L^p$.*

Proof: We have

$$\phi(\epsilon A)f = \int_0^\infty K_\epsilon(x,x')f(x')dx'. \tag{27}$$

We may assume without loss of generality that [C] and (25) hold for $\epsilon = 1$; when ϵ is sufficiently small, we may substitute $C \to \frac{1}{\epsilon}C$ without changing the value of the integral, since since K_ϵ does not depend on the particular contour satisfying [C]. Hence,

$$\begin{aligned} K_\epsilon(x,x') &= \frac{-1}{2\pi i}\int_{\frac{1}{\epsilon}C} \phi(\epsilon z) G_{-\frac{1}{z}}(x,x')dz \\ &= \frac{-1}{2\pi i}\int_C \frac{\phi(z)}{z}\frac{z}{\epsilon} G_{-\frac{\epsilon}{z}}(x,x')dz \\ &= g_\epsilon^{(1)}(x-x') + g_\epsilon^{(2)}(x+x') + R_\epsilon(x,x'), \end{aligned} \tag{28}$$

where (in the notation of (12))

$$g_\epsilon^{(i)}(x,x') = \frac{-1}{2\pi i}\int_C \frac{\phi(z)}{z}\frac{z}{\epsilon} G^{(i)}_{-\frac{\epsilon}{z}}(x,x')dz, \tag{29}$$

and R_ϵ is the remainder. By (15) and (iii), the mixed p,q norm

$$\|R_\epsilon(x,x')\|_{p,q} \underset{\epsilon\to 0}{\to} 0 \qquad (1 \le p,q \le \infty). \tag{30}$$

Hence, if $f(x) \in L^p(\mathrm{R}^+)$ $(0 \le p \le \infty)$, then for $\frac{1}{p} + \frac{1}{p'} = 1$

$$\begin{aligned} \|\int_0^\infty R_\epsilon(x,x')f(x')dx'\|_q &\le \int_0^\infty \|R_\epsilon(x,x')\|_{q,x} f(x')dx' \\ &\le \|R_\epsilon(x,x')\|_{q,p'}\|f\|_p \underset{\epsilon\to 0}{\to} 0. \end{aligned} \tag{31}$$

The function $g_\epsilon^{(1)}(x)$ is a convolution kernel which is symmetric and monotone decreasing in $|x|$. By the Minkowski inequality and (iii) of [C], $g_\epsilon^{(1)}(x) \in L^1(\mathrm{R}^+)$.

By the arguments in the proof of Theorem 1, this guarantees that for $f \in L^p(\mathrm{R}^+)$, $g_\epsilon^{(1)} * f(x) \underset{\epsilon\to 0}{\to} f(x)$ pointwise for x in the Lebesgue set of f, and in L^p $(1 \le p < \infty)$. It thus remains to prove that $\int_0^\infty g_\epsilon^{(2)}(x,x')f(x,x')dx' \to_{\epsilon\to 0} 0$ in L^p $(1 \le p \le \infty)$. This fact follows and from arguments identical to those proving (13). ∎

The hypotheses of the Theorem are complicated for the sake of generality. We now sacrifice generality for more direct applicability:

COROLLARY: Let $\phi(z)$ be analytic in an open domain D containing the closed set $\{z : |\arg z| \leq \gamma\}$ for some $\gamma > 0$, and $|\phi(z)| = O\Big(|z|^{-(1+\delta)}\Big)$ for some $\delta > 0$ as $z \to \infty$ in D. Let $\epsilon \to 0$ in the sector $|\arg \epsilon| \leq \gamma_1 < \frac{\gamma}{2}$. Then any $f \in L^2(\mathrm{R}^+)$ is ϕ-summable in L^p $(1 \leq p < \infty)$ and on its Lebesgue set as $\epsilon \to 0$.

The proof of Theorem 5 also carries over (with some simplifications) to the case of non-singular Sturm-Liouville expansions on a finite interval $[a, b]$; this situation is the focus of the classical summability theory for eigenfunction expansions. For brevity we present this result in a more restricted situation.

THEOREM 6: Let $q(x)$ be a real-valued continuous function on $[a, b]$, and $A = -\frac{d^2}{dx^2} + q$ have spectrum in $[-b, \infty]$. If $\phi(z)$ is analytic in a neighborhood D of $B_\gamma = \{z : |\arg z| \leq \gamma\}$, and $\phi(z) = 0(z^{-(1+\delta)})$ in D, then for $f \in L^2[a, b]$, the expansion of f in eigenfunctions of A is ϕ- summable on the Lebesgue set of A and in all $L^p[a, b]$ $(p < \infty)$ which contain f.

Among the classical summability methods encompassed by Theorems 5 and 6 are the Abel ($\phi(z) = e^{-cz}$) and Gauss-Weierstrass ($\phi(z) = e^{-cz^2}$) methods.

§ 4. Applications to Stable Summability

Stable summability has been a primary focus in the study of summability of eigenfunction expansions over the last twenty years (see [1,2,3,6]). It has practical applications in the theory of ill-posed problems in that small perturbations of eigenfunction expansions arise in practice from limitations on measurement.

If the ϕ summability method has summability means $\phi(\epsilon A)f(x)$, then the method is *pointwise stable* if there exists a non-trivial scaling $\gamma(\epsilon)$ such that for a net $\{f_\gamma\}$ of

L^2 functions with the property $\|f_\gamma - f\|_2 \leq \gamma$, $\phi(\epsilon A) f_{\gamma(\epsilon)}(x) \underset{\epsilon \to 0}{\longrightarrow} f(x)$ on the Lebesgue set of f.

The following theorem is proved in [1] on the semi-infinite interval, and the proof extends with little change to finite intervals.

THEOREM 7: *Suppose that $\phi(\lambda)$ is a real-valued function and $f, f_\gamma \in L^2$ on $\mathbf{R}^+$ or $[a, b]$ such that the following hold:*

(a) $\|f - f_\gamma\|_2 \leq \gamma$

(b) $\int_0^\infty \frac{\phi^2(\lambda)}{\sqrt{\lambda}} < \infty$

(c) *$\phi(\epsilon A) f(x) \underset{\epsilon \to 0}{\longrightarrow} f(x)$ uniformly on a bounded subset E of $\mathbf{R}^+$ or $[a,b]$.*

(d) *α is a function of γ such that as $\gamma \to 0, \alpha \to 0$, and $\frac{\gamma}{\alpha^{\frac{1}{4}}} \to 0$.*

Then $\phi(\epsilon A) f_{\gamma(\epsilon)}(x) \underset{\epsilon \to 0}{\longrightarrow} f(x)$ uniformly on E; in particular $\phi(\epsilon A)$ is a stable summation method.

Hypothesis (c), *a priori* summability, has been shown to hold for $\phi(\lambda)$ satisfying the hypothesis of Theorems 5 and 6; by Theorem 7, these theorems automatically provide a broad class of methods for which stable summability can be shown to hold. In [6], Tikhonov proved stable summability for $\phi(\lambda) = (1 + \lambda)^{-1}$ on the set of continuity points of $f \in L^2[a, b]$, with scaling $\frac{\gamma}{\alpha^{\frac{1}{2}}} \to 0$. This result is improved and generalized in several ways by Theorems 7 and 6.

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THE STIELTJES KERNEL AND SUMMING STURM-LIOUVILLE EIGENFUNCTION EXPANSIONS

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ABSTRACT: Given a general singular Sturm-Liouville (S-L) system on a semi-infinite domain possessing a discrete negative and continuous positive spectrum, the problem of inverting the generalized Fourier transform of $L^p(0, \infty)$ $(1 \leq p < \infty)$ functions is considered. We determine conditions under which the generalized Sturm-Liouville eigenfunction expansions of $L^p(0, \infty)$ $(1 \leq p < \infty)$ functions f are Stieltjes summable to f with respect to the $L^p(0, \infty)$ norm for $1 \leq p < \infty$ and pointwise on the Lebesgue set of f. The Stieltjes kernel, associated with the summability means, is defined to be a scaled Green's function associated with a perturbation of our S-L system. As an immediate application we see that the Stieltjes summability means of eigenfunction expansions with perturbed coefficients converge pointwise to the original function.

*Research partially supported by Air Force Office of Scientific Research grant 80-NM-279 and Army Research Office grant DAA G29-81-G-0011.

§1. Introduction: The basic question considered in this paper is given the generalized Fourier transform $F(\lambda)$ of an $L^p(0, \infty)$ $(1 \leq p < \infty)$ function f, how do we obtain f back again from $F(\lambda)$? Difficulties in answering this question stem from the generalized nature of our eigenfunctions, that $F(\lambda)$ need not be integrable, and for $p > 2$, $F(\lambda)$ may not be a function.

The answer to this question for Fourier and Sturm-Liouville eigenfunction expansions lies in summability theory. Harmonic analysis has used Cesáro, Abel, Gauss and in general, summability methods whose associated kernels are L'-dilations of radially decreasing convolution kernels on the n-dimensional Euclidean space $\mathbb{R}^n$ $(n \geq 1)$, $\mathbb{R}$ real ([1], [2], [3], [4]). From the perspective of self-adjoint differential operators Reisz (equivalent to Cesáro) summability has been applied to regular and singular Sturm-Liouville eigenfunction expansions ([5], [6], [7]).

Our motivation for considering Stieltjes summability lies in the fact that this is the method which naturally arises from Tikhonov's regularization principle for solving a class of ill-posed problem [8]. Stieltjes summability was introduced [9] for $L^2(0, \infty)$ functions in order to recover $f(x_0)$ from a singular Sturm-Liouville expansion in which the coefficients are slightly perturbed in the $L^2(0, \infty)$ norm. In this paper we consider a singular Sturm-Liouville (S-L) system defined on the half line with general boundary conditions and which possesses a discrete negative and continuous positive spectrum.

After providing the mathematical setting of our problem, we identify the Stieltjes kernel associated with the Stieltjes summability means. This kernel is a scaled Green's function where the Green's function is associated with a perturbation of our S-L system. Due to the nature of the problem, the eigenfunctions and hence the Stieltjes kernel have general representations. Thus the kernel does not have the computational character of kernels such as the Fejér and Poisson (which are the kernels associated with the Cèsaro and Abel summability methods). However we are able to show that our kernel possesses properties analogous to those of the convolution kernels.

Our principal result for $L^p(0, \infty)$ $(1 \leq p < \infty)$ functions f is the $L^p(0, \infty)$ $(1 \leq p < \infty)$ and pointwise convergence on the Lebesgue set of f of the Stieltjes summability means of the S-L expansion to f. We prove these summability results by using well known results in harmonic analysis for dilations of a convolution kernel [4]. We also use the fact that for $p = 2$, pointwise convergence of the Stieltjes means was proved [9] solely on the assumption that f belongs to $L^2(0, \infty)$. (We mention that we could give an alternate proof of the convergence of the Stieltjes means, with respect to the L^p-norm and on the set of continuity points of f, by using the properties of the Stieltjes kernel (Theorem 1) and classical techniques à la Titchmarsh.)

Finally, as an application we show that if the coefficients in the expansion of an $L^p(0, \infty)$ $(1 \leq p < \infty)$ function are perturbed slightly in the $L^2(0, \infty)$ norm, then the Stieltjes summability method is stable. That is, this method recovers a good approximation to f at points where f is sufficiently regular.

Acknowledgement. It is a pleasure to thank Professor Aileen Bonami of the University of Paris and Professor Mark Kon of Boston University who suggested the techniques used.

§2. The S-L System: A detailed mathematical formulation of our S-L system is now presented.

Let S-L denote the singular Sturm-Liouville system

$$u''(x) - q(x)u(x, \lambda) = -\lambda u(x, \lambda) \tag{1}$$

with boundary conditions

$$u(0, \lambda)\cos\beta + u'(0, \lambda)\sin\beta = 0 \quad \text{and} \quad u(\infty, \lambda) < \infty, \tag{2}$$

where $q(x) \in L'(0, \infty) \cap L^{\infty}(0, \infty)$ is continuous and real valued. The function $u(x, \lambda)$ (for all λ in the spectrum) are normalized by the conditions

$$u(0, \lambda) = \sin\beta \quad \text{and} \quad u'(0, \lambda) = -\cos\beta \tag{3}$$

The spectrum of S-L is bounded from below, discrete for $\lambda \le 0$ and continuous for $\lambda > 0$ ([7], Theorem 3.1, p. 209 and Theorem 3.2, p. 211). The non-positive spectrum is denoted by $\{\lambda_n\}$ and the associated eigenfunctions by $\{u(x, \lambda_n)\}$. In general $u(x, \lambda)$ denotes the eigenfunction associated with the spectral element λ, where $\lambda \in \{\lambda_n\} \cup (0, \infty)$.

For $f(x) \in L^2(0, \infty)$, the S-L expansion of f is given by

$$f(x) \sim \sum_{\lambda_n} F(\lambda_n)u(x, \lambda_n)d + \int_0^\infty F(\lambda)u(x, \lambda)d\rho(\lambda) = \int_{-b}^\infty F(\lambda)u(x, \lambda)d\rho(\lambda) \tag{4a}$$

where

$$F(\lambda) \sim \int_0^\infty f(x)u(x, \lambda)dx \tag{4b}$$

In (3a), $-b = \inf \lambda_n$, $\rho(\lambda)$ is the spectral function of the system under the normalization (3), and $d_n = \rho(\lambda_n^+) - \rho(\lambda_n^-)$. In (4b), $F(\lambda)$ is the generalized Fourier transform of $f(x)$ and $F \in L_\rho^2(-b, \infty)$ where L_ρ^2 denotes the square norm with respect to the measure $\rho(\lambda)$. The symbol $\sim$ denotes convergence in the L^2-norm as the upper limit of summation or integration becomes infinite.

When $f \in L^2(0, \infty)$ the expansion in (4a) converges to f in the $L^2(0, \infty)$ norm, but the convergence need not be pointwise. For $f \in L^p(0, \infty)$, $p \geq 1$ and $p \neq 2$, the question of existence of the generalized integrals in (4a) and (4b) must be determined. To illustrate the complexity of this question, we cite the classical Fourier integral. Every $f \in L^p(-\infty, \infty)$ $(1 \leq p \leq \infty)$ has a Fourier transform $F(\lambda)$ (defined as a tempered distribution) that coincides with an L^p function if $1 \leq p \leq 2$. But for $p > 2$ there exist L^p functions whose Fourier transforms cannot be expressed as a function. So one of our main objectives is to determine when (4b) and Stieltjes summability means of (4a) exist.

All proofs herein are for the class of singular continuous spectrum S-L expansions associated with the system (1) thru (3). We remark that these proofs carry over, more simply, to regular S-L expansions on finite intervals.

§3. The Stieltjes Kernel: The summator or weight function for the Stieltjes summability method is $\phi(\lambda) = (1 + \lambda)^{-1}$. The Stieltjes summability means, where they exist, of the S-L expansion (4a) are denoted by

$$S_\alpha(f; x) = \int_{-b}^{\infty} \frac{F(\lambda)}{1 + \alpha\lambda} u(x, \lambda) d\rho(\lambda) \qquad 0 < \alpha < \frac{1}{b} \tag{5}$$

where α is the summation parameter and x is fixed. The S-L expansion (4a) is called Stieltjes-summable at x_0 if $\lim_{\alpha \to 0} S_\alpha(f; x_0)$ exists and called Stieltjes-summable to f at x_0 if $\lim_{\alpha \to 0} S_\alpha(f; x_0) = f(x_0)$.

Formally $S_\alpha(f; x)$ may be rewritten as

$$S_\alpha(f; x) = \int_0^\infty f(x)\, K_\alpha(x, s) ds \tag{6}$$

where the kernel $K_\alpha(x, s)$ is formally given by

$$K_\alpha(x, s) = \int_0^\infty \frac{u(x, \lambda)u(s, \lambda)}{1 + \alpha\lambda}\, d\rho(\lambda).$$

The immediate objectives are to determine when $S_\alpha(f; x)$ is defined and to derive a formula for the kernel $K_\alpha(x, s)$.

It is a consequence of [9] that for $f \in L^2(0, \infty)$

$$S_\alpha(f; x) = \frac{1}{\alpha} \int_0^\infty G(x, s; \tfrac{1}{\alpha})\, f(s)ds \tag{7}$$

where $G(x, s; \frac{1}{\alpha})$ is the Green's function of the distributional equation

$$-u''(x, \lambda) + [q(x) + \tfrac{1}{\alpha}]u(x, \lambda) = -\,\delta_s(x) \tag{8}$$

where $u(0) = 0$ and $u(\infty) < \infty$, and where δ_s is the Dirac distribution centered at s. From (6) and (7), it is natural to define the Stieltjes kernel $K_\alpha(x, s)$ for $f \in L^2(0, \infty)$ to be $\frac{1}{\alpha} G(x, s; \frac{1}{\alpha})$ where $G(x, s; \frac{1}{\alpha})$ is as above.

To determine under what conditions (7) *would be true for* $L^p(0, \infty)$ functions, $(1 \le p < \infty)$, we must, due to the nature of the eigenfunctions, consider cases based on whether $\sin \beta = 0$ or not. In general, however, the procedure is as follows: For an $L^1(0, \infty)$ function $f^{(1)}$, we let $\{f_n\}_n$ be a sequence of functions which belongs to a dense subset of $L^1(0, \infty)$, say $L^1 \cap L^2$, and which converges to $f^{(1)}$ with respect to the L^1-norm. We then prove identity (7) for $L^1(0, \infty)$ functions by using (7) for $L^2(0, \infty)$ functions. An interpolation theorem will prove (7) for $L^p(0, \infty)$ $(1 < p < 2)$ functions. Lastly, for $p > 2$ we impose restrictions to insure that the improper integrals exist and thereby proving the identity (7) by standard arguments.

First we need a bound on the Green's function $G(x, s; \frac{1}{\alpha})$. Since we are unable to calculate $G(x, s; \frac{1}{\alpha})$ directly, we calculate the Green's function $G^*(x, s; \frac{1}{\alpha})$ for $\alpha > 0$ of

$$-u''(x) + \frac{1}{\alpha} u(x) = - \delta_s(x) \tag{9}$$

where

$$u(0) = \sin \beta \quad \text{and} \quad u(\infty) < \infty .$$

$$G^*(x, s; \tfrac{1}{\alpha}) = \begin{cases} \dfrac{\sqrt{\alpha}\; e^{-x/\sqrt{\alpha}}}{\sqrt{\alpha}\cos\beta - \sin\beta}\left\{\sqrt{\alpha}\cos\beta\sinh\dfrac{s}{\sqrt{\alpha}} - \sin\beta\cosh\dfrac{s}{\sqrt{\alpha}}\right\}, & s < x \\[2ex] \dfrac{\sqrt{\alpha}\; e^{-s/\sqrt{\alpha}}}{\sqrt{\alpha}\cos\beta - \sin\beta}\left\{\sqrt{\alpha}\cos\beta\sinh\dfrac{x}{\sqrt{\alpha}} - \sin\beta\cosh\dfrac{x}{\sqrt{\alpha}}\right\}, & s > x \end{cases} \tag{10}$$

We recall that the $q(x)$ in our S-L system is uniformly bounded, say by $M > 0$. This was done in order to be able to bound $G(x, s; \frac{1}{\alpha})$. The import of our next lemma is that the Green's function of our S-L system is bounded between a combination of Green's functions, each of which is bounded by a bell-shaped convolution kernel which is radially decreasing in the parameter α on IR.

<u>Lemma 1</u>: (a) $G^*(x, s; \frac{1}{\alpha}) = G^{(1)}(x, s; \frac{1}{\alpha}) + G^{(2)}(x, s; \frac{1}{\alpha})$ (11a)

where

$$G^{(1)}(x, s; \tfrac{1}{\alpha}) = \sqrt{\alpha}\,\frac{e^{-|x-s|/\sqrt{\alpha}}}{2}, \quad G^{(2)}(x, s; \tfrac{1}{\alpha}) =$$

$$= \frac{\sqrt{\alpha}}{2}\,\frac{\sin\beta + \sqrt{\alpha}\cos\beta}{(\sin\beta - \sqrt{\alpha}\cos\beta)}\, e^{-|x+s|/\sqrt{\alpha}} \qquad 0 \le x, s, < \infty \tag{11b}$$

(b) For $|q(x)| \le M$, $-M + \frac{1}{\alpha} > 0$ and $x \in (0, \infty)$

$$G^{**}(x, s; M + \tfrac{1}{\alpha}) = G^{(1)}(x, s; M + \tfrac{1}{\alpha}) - |G^{(2)}(x, s; M + \tfrac{1}{\alpha})| \le G(x, s; \tfrac{1}{\alpha})$$

$$\le G^{(1)}(x, s; -M + \tfrac{1}{\alpha}) + |G^{(2)}(x, s; -M + \tfrac{1}{\alpha})| = G^{**}(x, s; -M + \tfrac{1}{\alpha}) \tag{12}$$

(c) $$\frac{1}{\alpha}\,|G(x, s; \tfrac{1}{\alpha})| \le \frac{3}{\sqrt{\alpha}}\, e^{-|x-s|/\sqrt{\alpha}} \tag{13}$$

Proof. Part (a) follows from an easy calculation on (10). Part (b) follows by substituting $\pm M + \frac{1}{\alpha}$ for $\frac{1}{\alpha}$ in (11b) and comparing terms. Part (b) implies part (c).

The following approximations on the eigenfunctions and spectral function ρ for large λ follows from [7, Equation 3.5, p. 205] and [7, Theorem 3.2, p. 211 and p. 206] respectively.

Lemma 2: As $\lambda \to \infty$, $u(x, \lambda)$ and $\rho(\lambda)$ satisfy

(a) if $\sin \beta = 0$

$$
\begin{aligned}
u(x, \lambda) &= -\frac{\cos \beta}{\sqrt{\lambda}} \sin\sqrt{\lambda}\, x + O(\tfrac{1}{\lambda}) \\
\rho'(\lambda) &= \frac{\sqrt{\lambda}}{\pi \cos^2 \beta} + O(1),
\end{aligned}
\tag{14}
$$

and

(b) if $\sin \beta \neq 0$

$$
\begin{aligned}
u(x, \lambda) &= \sin \beta \cos\sqrt{\lambda}\, x + O(\tfrac{1}{\lambda}) \\
\rho'(\lambda) &= \frac{1}{\pi \sqrt{\lambda} \sin^2 \beta} + O(\tfrac{1}{\lambda}).
\end{aligned}
\tag{15}
$$

We remark that $u(x, \lambda)$ is bounded in x for fixed λ, and bounded in λ for fixed x, but it is not bounded jointly. Moreover, $u(x, \lambda)$ may get large as λ gets small for x large.

We prove the lemmas in the following order: $\sin \beta \neq 0$ and $1 \le p \le 2$, $\sin \beta = 0$ and $1 \le p \le 2$; and lastly for arbitrary $\sin \beta$ and $p > 2$. We have noticed that the proofs of these lemmas go through identically if we replace the Stieltjes weight function $(1 + \lambda)^{-1}$ by a summator function $\phi(\lambda)$ which is analytic, bounded and such that $\int_{-b}^{\infty} \frac{\phi(\lambda)}{\sqrt{\lambda}} d\lambda < \infty$. Moreover, if our summator function $(1 + \lambda)^{-1}$ were replaced by a function $\phi(\lambda)$ which is in $L^1(0, \infty)$ and analytic, then the following lemmas can be obtained with less assumptions. (Of course, the Stieltjes weight function satisfies $\int_{-b}^{\infty} \frac{\phi(\lambda)d\lambda}{\sqrt{\lambda}} < \infty$, but is not in $L'(0, \infty)$.)

In Lemma 3 the following notation is used when $\{f_n\}_{n=1}^{\infty}$ *is a sequence of* $L^1 \cap L^2$ functions. Let

$$F_n(\lambda) = \int_0^{\infty} f_n(x)u(x, \lambda)dx$$

and

$$S_\alpha(f_n; x) = \int_{-b}^{\infty} \frac{F_n(\lambda)}{1 + \alpha\lambda} u(x, \lambda)d\rho(\lambda)$$

Lemma 3. For $\sin \beta \neq 0$, if $f \in L^1(0, \infty)$ and $\{f_n\}_{n=1}^{\infty}$ is a sequence of $L^1 \cap L^2$ functions which converges to f in the L^1-norm, and if $u(x, \lambda)$ is uniformly bounded in λ for x large, then

(a) $F_n(\lambda) - F(\lambda)$ converges uniformly in λ to zero as $n \to \infty$;

(b) $S_\alpha(f_n; x) - S_\alpha(f; x)$ converges pointwise to zero as $n \to \infty$;

(c) $S_\alpha(f; x)$ is continuous in x and belongs to $L^1(0, \infty)$;

(d) $S_\alpha(f; x) = \frac{1}{\alpha} \int_0^{\infty} G(x, s; \frac{1}{\alpha})f(s)ds$ pointwise and where $G(\cdot\, ; \frac{1}{\alpha})$ is Green's function for (8).

Proof: By Lemma 2 part (b), $u(x, \lambda)$ is uniformly bounded in λ for bounded x. The proof of part (a) follows by our assumption on $u(x, \lambda)$ and $||f_n - f||_1 \to 0$ as $n \to \infty$.

To prove part (b), we write

$$S_\alpha(f_n; x) - S_\alpha(f; x) = \int_{-b}^{\infty} [F_n(\lambda) - F(\lambda)] \frac{u(x, \lambda)}{1 + \alpha\lambda} d\rho(\lambda).$$

In the case $\sin \beta \neq 0$, Lemma 2 gives $u(x, \lambda) = O(\sqrt{\lambda})$, $d\rho(\lambda) = O(\frac{1}{\lambda})$ and so the product of the last three terms in the integral is $O(\frac{1}{\lambda^{3/2}})$. The proof is completed by using part (a).

To prove part (c) one need only observe that $F(\lambda) \varepsilon L^{\infty}(0, \infty)$ (as $f \varepsilon L'(0, \infty)$), $u(x, \lambda)$ is uniformly bounded for bounded x-intervals, and $\frac{d\rho(\lambda)}{1 + \alpha\lambda} = O(\frac{1}{\lambda^2})$. Thus the improper integral $\int_{-b}^{\infty} \frac{F(\lambda)}{1 + \alpha\lambda} u(x, \lambda) d\rho(\lambda)$ converges absolutely and uniformly in x. Hence $S_\alpha(f; x)$ is continuous in x and belongs to $L'(0, \infty)$.

Finally to prove part (d), we recall [9] that for $f_n \varepsilon L^2(0, \infty)$

$$S_\alpha(f_n; x) = \frac{1}{\alpha} \int_0^{\infty} G(x, s; \tfrac{1}{\alpha}) f_n(s) ds.$$

pointwise. Next we note that

$$\frac{1}{\alpha} \int_0^{\infty} G(x, s; \tfrac{1}{\alpha}) f_n(s) ds \text{ converges pointwise to } \frac{1}{\alpha} \int_0^{\infty} G(x, s; \tfrac{1}{\alpha}) f(s) ds$$

as

$$\left| \frac{1}{\alpha} \int_0^{\infty} G(x, s; \tfrac{1}{\alpha}) [f_n(s) - f(x)] ds \right| \le \frac{1}{\alpha} \| G(\cdot\, ; \tfrac{1}{\alpha}) \|_\infty \, \| f_n - f \|_1 \to 0$$

pointwise as $n \to \infty$. (For the bound on $G(\cdot\, ; \frac{1}{\alpha})$ see Lemma 1).

So $S_\alpha(f_n; x)$ converges pointwise to both $S_\alpha(f; x)$ (part (b)) and $\frac{1}{\alpha} \int_0^{\infty} G_\alpha(x, s) f(s) ds$ and so the proof of part (d) is complete.

Next we use an interpolation theorem to extend the result of part (d) to $L^p(0, \infty)$ functions for p between 1 and 2.

<u>Lemma 4</u>: For $\sin \beta \neq 0$, $f \varepsilon L^p(0, \infty)$, $1 < p < 2$ and under the assumptions of Lemma 3, then

(a) $S_\alpha(f; x)$ is continuous in x;

(b) $S_\alpha(f; x) = \frac{1}{\alpha} \int_0^{\infty} G(x, s; \frac{1}{\alpha}) f(s) ds$ pointwise and where $G(\cdot\, ; \frac{1}{\alpha})$ is the Green's function for (8);

(c) $S_\alpha(f; x)$ belongs to $L^p(0, \infty)$, $1 < p < 2$.

Proof: If $f \in L^p(0, \infty)$ where $1 < p < 2$, then f can be expressed as the sum of an $L^1(0, \infty)$ and $L^2(0, \infty)$ function, namely

$$f(x) = f(x)\chi_{|f|\le 1}(x) + f(x)\chi_{|f|\ge 1}(x) = f_1(x) + f_2(x).$$

We then write

$$S_\alpha(f; x) = S_\alpha(f_1; x) + S_\alpha(f_2; x) = \int_0^\infty \frac{F(\lambda)}{1 + \alpha\lambda} u(x, \lambda)d\rho(\lambda)$$

where $F(\lambda)$ is the sum of the $L^\infty(0, \infty)$ Fourier coefficient $F_1(\lambda)$ associated with f_1 and the $L^2(0, \infty)$ Fourier coefficient $F_2(\lambda)$ associated with f_2.

The continuity of $S_\alpha(f; x)$ now follows by separate arguments for the continuity of $S_\alpha(f_1; x)$ and $S_\alpha(f_2; x)$. The first follows by Lemma 3 and the second [9] by Cauchy-Schwarz and [7, Corollary 1, p. 116].

Similarly, the proof of part (b) follows by separate arguments on $S_\alpha(f_1; x)$ and $S_\alpha(f_2; x)$ (Lemma 3 and [9]).

Finally we show that $S_\alpha(f; x)$ belongs to $L^p(0, \infty)$ by observing that $G(x, s; \frac{1}{\alpha})$ is bounded by half of a bell-shaped convolution kernel (see Lemma 1). As $f \in L^p(0, \infty)$, $G(x, s; \frac{1}{\alpha})$ and its bound $\in L^1(0, \infty)$, the convolution is well defined and converges absolutely a.e. in x. So by Young's inequality we have

$$||S_\alpha(f, s)||_p = ||\frac{1}{\alpha}\int_0^\infty G(x, s; \frac{1}{\alpha})f(s)ds||_p \le C||\int_0^\infty \frac{e^{-|x-s|/\sqrt{\alpha}}}{\sqrt{\alpha}} f(s)ds||_p$$

$$\le C||\frac{e^{-|x-s|/\sqrt{\alpha}}}{\sqrt{\alpha}}||_1 \; ||f||_p, \quad \text{where} \quad C = 3.$$

We will use the following notation in the next two lemmas. Let $L^2_{loc}(0, \infty)$ denote the space of functions which are square integrable over every compact subset of $(0, \infty)$. For $f \in L^p(0, \infty)$ we define

$$f_N(x) = \int_{-b}^N F(\lambda)\, u(x, \lambda)d\rho(\lambda)$$

and

$$S_\alpha(f_N, x) = \int_{-b}^{N} F(\lambda) \frac{u(x, \lambda)}{1 - \alpha\lambda} d\rho(\lambda),$$

where $F(\lambda) \in L^\infty[-b, N]$ for $p = 1$, and $F(\lambda) \in L^2_{loc}(0, \infty)$ for $p > 2$.

We continue now with the case $\sin \beta = 0$ and $1 \le p \le 2$.

Lemma 5: For $\sin \beta = 0$ and $f \in L^1(0, \infty)$, if

$$f_N(x) \to f(x) \text{ in } L^1(0, \infty),$$

and

$$S_\alpha(f_N; x) \to S_\alpha(f; x) \text{ pointwise as } N \to \infty$$

then

(a) $S_\alpha(f_N; x) = \frac{1}{\alpha} \int_0^\infty G(x, s; \frac{1}{\alpha}) f_N(s) ds$

(b) $S_\alpha(f; x) = \frac{1}{\alpha} \int_0^\infty G(x, s; \frac{1}{\alpha}) f(s) ds$

(c) $S_\alpha(f; x) \in L^1(0, \infty)$.

Proof. Define

$$F_N(\lambda) = F(\lambda) \chi_{[-b,N]}(\lambda)$$

and

$$F_{N,\alpha}(\lambda) = \frac{F(\lambda)}{1 + \alpha\lambda} \chi_{[-b,N]}(\lambda) .$$

Clearly F_N and $F_{N,\alpha}$ belong to $L^2 \cap L^\infty$. This in turn implies that f_N and $S_\alpha(f_N; x)$ belong to $L^2(0, \infty)$. And so by [9]

$$S_\alpha(f_N; x) = \frac{1}{\alpha} \int_0^\infty G(x, s; \frac{1}{\alpha}) f_N(s) ds.$$

To prove part (b) we observe that $\frac{1}{\alpha} \int_0^\infty G(x, s; \frac{1}{\alpha}) f_N(s) ds$ converges to $\frac{1}{\alpha} \int_0^\infty G(x, s; \frac{1}{\alpha}) f(s) ds$ pointwise as $||f_N - f||_1 \to 0$ as $N \to \infty$. By hypotheses, $S_\alpha(f_N; x)$ converges pointwise to $S_\alpha(f; x)$, and so the equality holds.

Part (c) follows from (b), a bound on G and $f \in L^1(0, \infty)$.

Again we use an interpolation theorem to extend Lemma 5 part (b) to $L^p(0, \infty)$ functions, $1 < p < 2$, when $\sin \beta = 0$.

<u>Lemma 6</u>: For $\sin \beta = 0$, $f \in L^p(0, \infty)$, $1 < p < 2$ and under the assumptions of Lemma 5, then

(a) $S_\alpha(f; x) = \frac{1}{\alpha}\int_0^\infty G(x, s; \frac{1}{\alpha})f(x)ds$

(b) $S_\alpha(f; x) \in L^p(0, \infty)$, $1 < p < 2$.

<u>Proof</u>: If $f \in L^p(0, \infty)$, $1 < p < 2$, then as in the proof of Lemma 4, f can be expressed as the sum of an $L^1(0, \infty)$ and an $L^2(0, \infty)$ function, f_1, f_2 respectively. Thus the generalized Fourier coefficient of f belongs to $L^\infty(0, \infty) + L^2(0, \infty)$. Clearly

$$S_\alpha(f_N; x) = \int_b^N F(\lambda) \text{ is } (x, \lambda)d\rho(\lambda)$$

exists for finite N. Letting $S_\alpha(f; x) = S_\alpha(f_1; x) + S_\alpha(f_2; x)$ and applying Lemma 5 to $S_\alpha(f_1; x)$ and (7) to $S_\alpha(f_2; x)$ we see part (a) is proved.

The proof of part (b) is similar to that of Lemma 4 part (c).

Lastly we consider the case of $L^p(0, \infty)$ functions for $p > 2$ and $\sin \beta$ arbitrary. The assumptions of the following lemma are motivated by the fact that for $p > 2$, $F(\lambda)$ may not be a function.

<u>Lemma 7</u>: For $f \in L^p(0, \infty)$, $p > 2$, if

(1) $f_N(x) = \int_{-b}^N F(\lambda)u(x, \lambda)d\rho(\lambda)$ exists, converges pointwise for each N and for each fixed x where $F(\lambda) \in L^2_{loc}(0, \infty)$,

(2) $f_N(x) \to f(x)$ in $L^p(0, \infty)$, $p > 2$;

(3) $S_\alpha(f_N; x)$ exists for each fixed N and x, and converges pointwise to $S_\alpha(f; x)$ for each fixed α, x as $N \to \infty$,

then

(a) $S_\alpha(f;\, x) = \frac{1}{\alpha}\int_0^\infty G(x,\, s;\, \frac{1}{\alpha})f(s)ds;$

(b) $S_\alpha(f;\, x) \in L^p(0,\, \infty),\ p > 2.$

Proof. As $F(\lambda) \in L^2_{loc}(0,\, \infty)$ implies $f_N \in L^2(0,\, \infty)$, we have

$$S_\alpha(f_N;\, x) = \int_{-b}^\infty F(\lambda)\ \chi_{[-b,N]}(\lambda)\ \frac{u(x,\, \lambda)}{1 + \alpha\lambda}\, d\rho(\lambda) = \frac{1}{\alpha}\int_0^\infty G_\alpha(x,\, s)f_N(s)ds.$$

Moreover

$$\left|\frac{1}{\alpha}\int_0^\infty G(x,\, s;\, \frac{1}{\alpha})f_N(s)ds - \frac{1}{\alpha}\int_0^\infty G(x,\, s;\, \frac{1}{\alpha})f(s)ds\right| \le \left|\left|\frac{1}{\alpha}G(\cdot\, ;\, \frac{1}{\alpha})\right|\right|_q\ ||f_N - f||_p \to 0$$

as $N \to \infty$ for $p > 0$, $q > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$. Thus $\frac{1}{\alpha}\int_0^\infty G(x,\, s;\, \frac{1}{\alpha})f_N(s)ds$ converges pointwise to $\frac{1}{\alpha}\int_0^\infty G(x,\, s;\, \frac{1}{\alpha})f(s)ds$. The proof of part (a) is completed by using hypothesis (3).

The proof of part (b) follows from Holder's inequality.

The following definition is now motivated by Lemmas 3 through 7.

Definition: The Stieltjes kernel $K_\alpha(x,\, s)$ for the Stieltjes means (5) of an $L^p(0,\, \infty)$, $p \le 1$, function f, is defined to be $\frac{1}{\alpha}\, G(x,\, s;\, \frac{1}{\alpha})$ where $G(x,\, s;\, \frac{1}{\alpha})$ *is the Green's function for the perturbed S-L system* (8).

Properties of $K_\alpha(x,\, s)$, the Stieltjes kernel, analogous to those of the convolution kernels are now proved.

Theorem 1: If $K_\alpha(x,\, s)$ is the Stieltjes kernel, then

(a) $\lim\limits_{\alpha\to 0}\int_0^\infty K_\alpha(x,\, s)ds = 1;$

(b) For each $\varepsilon > 0$, $\lim\limits_{\alpha\to 0}\int_{|x-s|>\varepsilon} K_\alpha(x,\, s)ds = 0,\ 0 \le x,\, s < \infty;$

(c) For each $\varepsilon > 0$, $\lim\limits_{\alpha\to 0}[\int_{|x-s|>\varepsilon} K^q_\alpha(x,\, s)ds]^{1/q}$ for $q > 1;$

(d) $k_\alpha(x, s) \to 0$ uniformly for all x and s as $\alpha \to 0$ for which $|x - s| > \varepsilon > 0$ and $0 \le x < s < \infty$.

Proof: By Lemma 1 it follows that

$$\frac{1}{\alpha}\int_0^\infty G^{**}(x, s; M + \frac{1}{\alpha})ds \le \frac{1}{\alpha}\int_0^\infty G(x, s; \frac{1}{\alpha})ds \le \frac{1}{\alpha}\int_0^\infty G^{**}(x, s; -M + \frac{1}{\alpha})ds.$$

An easy calculation shows that the upper and lower bounds approach one as $\alpha \to 0$. As $K_\alpha(x, s) = \frac{1}{\alpha} G_\alpha(x, s)$, part (a) is proved.

To establish part (b) we use Lemma 1 part (c) and evaluate the bounding function on $|x - s| > \varepsilon$ as $\alpha \to 0$.

Proof of part (c) is the same as (b).

Finally to prove part (d) we use Lemma 1 part (c) and note that the bounding function approaches zero as α approaches zero independent of x and s provided $|x - s| > \varepsilon > 0$.

§4. Recovering f from $F(\lambda)$: By using two standard theorems of harmonic analysis [4, Theorem 1.18, p. 10 and Theorem 1.25, p. 13], we now prove the principal result of this paper.

Theorem 2: Let $f \in L^p(0, \infty)$ $(1 \le p < \infty)$ and $K_\alpha(x, s)$ be the Stieltjes kernel. Then

(a) $$\frac{1}{\alpha}\int_0^\infty G(x, s; \frac{1}{\alpha})f(s)ds = \int_0^\infty K_\alpha(x, s)f(s)ds \to f(x) \text{ as } \alpha \to 0$$

in $L^p(0, \infty)$ $(1 \le p < \infty)$ and pointwise on the Lebesgue set of f;

(b) Under the hypotheses of Lemmas 3-7, the Stieltjes summability means

$$S_\alpha(f;\, x) = \int_{-b}^{\infty} \frac{F(\lambda)}{1 + \alpha\lambda}\, u(x,\, \lambda) f(s) ds \to f(x) \text{ as } \alpha \to 0,\ 0 < \alpha < \frac{1}{b}$$

in $L^p(0, \infty)$ $(1 \le p < \infty)$ and pointwise on the Lebesgue set of f.

Proof: Referring to Lemma 1, we write,

$$G(x,\, s;\, \tfrac{1}{\alpha}) = G^{**}(x,\, s;\, M + \tfrac{1}{\alpha}) + [G(x,\, s;\, \tfrac{1}{\alpha}) - G^{**}(x,\, s;\, M + \tfrac{1}{\alpha})]$$

$$\le G^{**}(x,\, s;\, M + \tfrac{1}{\alpha}) + [G^{**}(x,\, s;\, -M + \tfrac{1}{\alpha}) - G^{**}(x,\, s;\, M + \tfrac{1}{\alpha})].$$

Similarly,

$$G^{**}(x,\, s;\, -M + \tfrac{1}{\alpha}) + [G^{**}(x,\, s;\, M + \tfrac{1}{\alpha}) - G^{**}(x,\, s;\, -M + \tfrac{1}{\alpha})] \le G(x,\, s;\, \tfrac{1}{\alpha}).$$

Each of the G^{**} is expressed in terms of $G^{(1)}$ and $G^{(2)}$ (see (11 a-b) and (12) of Lemma 1).

Now extend f(x) to be 0 for $x \le 0$. We interpret the $G^{(1)}$ to be L^1-dilations of radially decreasing convolution kernels in IR. So the conclusion of this theorem holds for all $G^{(1)}$ by the well-known theorems in harmonic analysis. That is, $\frac{1}{\alpha}\int_0^\infty G^{(1)}(x,\, s; \pm M + \frac{1}{\alpha}) f(s) ds \to f(x)$ as $\alpha \to 0$ in $L^p(1 \le p < \infty)$ and pointwise on the Lebesgue set of f.

Considering $G^{(2)}$ on IR, replace s by -s. This changes $G^{(2)}(x,\, s;\, \pm M + \frac{1}{\alpha})$ into a convolution kernel on IR and f(s) into a function which is 0 on IR^+. And so $\frac{1}{\alpha}\int_0^\infty G^{(2)}(x,\, s;\, \pm M + \frac{1}{\alpha}) f(s) ds \to 0$ as $\alpha \to 0$ in $L^p(1 \le p < \infty)$ and on the Lebesgue set of f.

To complete the proof of part (a) we need only observe that

$$\frac{1}{\alpha}\int_0^\infty G^{**}(x,\, s;\, \pm M + \tfrac{1}{\alpha}) f(s) ds \to f(x) \text{ as } \alpha \to 0$$

and

$$\frac{1}{\alpha}\int_0^\infty [G^{**}(x,\, s;\, -M + \tfrac{1}{\alpha}) - G^{**}(x,\, s;\, M + \tfrac{1}{\alpha})] f(s) ds \to f(x) - f(x) = 0$$

as $\alpha \to 0$ in $L^p(1 \le p < \infty)$ and on the Lebesgue set of f.

The proof of part (b) is immediate after noting that under the conditions of Lemmas 3-7, $S_\alpha(f; x) = \int_0^\infty K_\alpha(x, s)f(s)ds = \frac{1}{\alpha}\int_0^\infty G_\alpha(x, s)f(s)ds$.

§5. Application to Stable Summability: In experiments which give the coefficients of eigenfunction expansions [8], measuring errors cause small perturbations in the expansion coefficients. Thus stable summability methods which recover from the perturbed expansions a good approximation to the original function f, at points where f is sufficiently regular, are of interest.

For the remainder of this paper, we assume that $f \in L^p(0, \infty)$ for $1 \le p < \infty$ satisfies the hypotheses of Lemmas 3-7. This insures that the Stieltjes summability means $S_\alpha(f; x)$ exist. As usual, $F(\lambda)$ denotes f's generalized Fourier transform. Let $\{f_\gamma(x)\}_\gamma$ be a sequence of $L^r(0, \infty)$ function $1 \le r < \infty$ which also satisfy Lemmas 3-7 and such that the associated generalized Fourier transforms $\{F_\gamma(\lambda)\}$ denote a net of approximations to $F(\lambda)$ such that to each value of the index γ, $F_\gamma(\lambda)$ satisfies

$$||F_\gamma - F||_2 = \{\int_{-b}^\infty |F_\gamma(\lambda) - F(\lambda)|^2 d\rho(\lambda)\}^{1/2} \le \gamma .$$

We say a summability method is pointwise stable if there exists a non-trivial scaling $\gamma(\alpha)$ such that for $\{F_\gamma(\lambda)\}_\gamma$ satisfying $||F_\gamma - F||_2 \le \gamma$ and $S_\alpha(f; x) \to f(x)$ pointwise as $\alpha \to 0$, then $S_\alpha(f_\gamma; x) \to f(x)$ pointwise as $\alpha \to 0$.

Our final result says that (under our restrictions) the Stieltjes summability means $S(f_\gamma; x)$ furnish a stable summation method, if the summation parameter α is approximately scaled to go to zero with γ. The proof of this theorem is essentially the same as [10, Theorem 1, p. 282].

Theorem 3: Let $f \in L^p(0, \infty)$ $(1 \le p < \infty)$ and $\{f_\gamma\}_\gamma \in L^r(0, \infty)$ $(1 \le r < \infty)$ satisfy the hypotheses of Lemmas 3-7. Suppose that the following hold

(1) $||F_\gamma - F||_2 \le \gamma$

(2) $S_\alpha(f; x) \to f(x)$ as $\alpha \to 0$ uniformly on a bounded subset E of $(0, \infty)$

(3) α is a function of γ such that both $\alpha \to 0$ and $\gamma/\alpha^{1/4} \to 0$ as $\gamma \to 0$. Then

(a) $S_\alpha(f; x) \to f(x)$ as $\alpha \to 0$ uniformly in E, and

(b) S_α *is a stable summability method*

We observe that this theorem holds for general summability methods where the summator function ϕ is real valued and $\int_0^\infty \frac{\phi^2(\lambda)dx}{\sqrt{\lambda}} < \infty$.

Finally, we comment that the Stieltjes kernel for $S_\alpha(f_\gamma; x)$ is the same as the Stieltjes kernel for $S_\alpha(f, x)$.

Remarks: We close this paper with two remarks. First all proofs herein were carried out for the class of singular continuous S-L expansions. We emphasize that the hypotheses of the results and their proofs are simplified for regular S-L expansions on finite intervals.

Second, in a joint paper with Professor Mark Kon, the results herein are extended to proving summability for a class of singular Sturm-Liouville expansions using dilation of analytic multipliers [12].

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